NOTES AND DISCUSSIONS

THE GENERATION OF NUMBERS IN PLATO'S *PARMENIDES*

At *Parmenides* 143A–B, Unity (or "the One") is distinguished from being (or its own being) and difference. At 143C–D, it is shown that these items are numberable; for if we pick out being and difference, or being and unity, or unity and difference, in each case we pick out both of two things; each of two is one; and when any one whatever is added to any pair whatever, the sum is three.

Parmenides next proceeds to argue, in 143D–44A, that if there is two there is twice, since two is twice one, and if three thrice, since three is thrice one. Therefore there is thrice two and twice three. Therefore there are even-times even numbers, odd-times odd numbers, even-times odd numbers, and odd-times even numbers. No number is then left remaining which does not necessarily exist, and the conclusion is drawn that there is an unlimited multitude of things which are.

Parmenides' account has often been taken as a generation or derivation of number, an interpretation that appears to be as old as Aristotle.[1] I propose here to argue, however, that in either of the two senses of number relevant to the discussion—number as a plurality of units, or as a property of such pluralities—this interpretation is mistaken.

Before proceeding, it will be well to recall that the Greek concept of number, *arithmos*, was restricted to the natural numbers exclusive of zero; it is not by accident that the verb *arithmein* means "to count." Euclid therefore defines number as "a multitude composed of units,"[2] and this, allowing for minor verbal differences, is typical of the Greek treatment generally.[3] This definition excludes fractions, surds, and zero as numbers,[4] and also implies that one itself is not a number, since it is not a multitude. Aristotle, reflecting the mathematical tradition of his time, held that one is not a number, but the beginning and measure of number,[5] and explained his meaning as follows:[6] "'the one' means the measure of some plurality, and 'number' means a measured plurality and a plurality of measures. (Thus it is natural that one is not a number; for the measure is not measures, but both the measure and the one are starting-points.)" Two, then, is the first number,[7] and it is by establishing the fact that the existence of Unity implies the existence of pluralities numerable by the first number, rather than by one, that Parmenides begins his account.

We may first inquire whether Parmenides' account may be construed as generating numbers as pluralities of units. It was F. M. Cornford's view that it might:[8] "From the simple consideration of 'One Entity,' with its two parts and the difference between them, we have derived the unlimited plurality of numbers. Each of the three terms is 'one entity' and can thus be treated as a unit; and by adding and multiplying these units we can reach any number (plurality of units), however great." This, however, is mistaken. If a given number is a plurality of units, then to derive that number is to prove that there are as many units as the number. The multiplication involved in "twice two"

1. See *Metaph.* 1. 987b32–34; see also H. F. Cherniss, *Aristotle's Criticism of Plato and the Academy*, I (Baltimore, 1944), n. 106; W. D. Ross, *Aristotle's "Metaphysics,"* I (Oxford, 1924), 175, and *The Works of Aristotle: "Metaphysica"* (Oxford, 1928), n. *ad loc.* A generation of numbers was also, presumably, a feature of the Neoplatonic interpretation of the passage, although the remaining fragment of Proclus' commentary here does not imply this (*Schol. in Parm.* 1261–62, Cousin). For Plotinus' discussion of number, see *Enn.* 6. 6.

2. *Elements* 7, Def. 2.

3. See T. L. Heath, *The Thirteen Books of Euclid's "Elements,"* II (New York, 1956), 280.

4. Rational fractions were handled as ratios of numbers, and surds as ratios of lengths.

5. *Metaph.* 5.1021a13; cf. 1016b18 ff.

6. *Ibid.* 14.1088a5–7, trans. Ross.

7. *Ibid.* 13.1085b10, and Cherniss, *op. cit.*, p. 303.

8. *Plato and Parmenides* (London, 1939), p. 141.

will require an existence assumption, namely, that four is a number, that is, that there are four units. It will also require exclusion of units: if four is a plurality of units, and $2 \times 2 = 4$, then the units in the first pair cannot be identical with units in the second; 4 cannot be, say, three units, one of which is counted twice. In short, if number is a plurality of units, it cannot be generated by the use of multiplication,[9] since the use of multiplication assumes the existence of pluralities corresponding to its products. Parmenides, then, working with only three units—unity, being, and difference—cannot produce pluralities greater than three by multiplication.

This difficulty, it may be added, cannot be avoided by treating the propositions of arithmetic as hypothetical. For Parmenides goes on to infer that since number, which is unlimited in multitude, has a share of being, each number must have a share of being, and that, therefore, the parts of being are unlimited in multitude (144A–C). This assumes, not *if* there are numbers, *then* they will be unlimited in multitude, but *that* there are numbers *and* they are unlimited in multitude. If numbers are pluralities of units which require generation, no basis for this inference has been provided.

The foregoing seems sufficient reason to suppose that Parmenides is not attempting to generate numbers as pluralities of units, but a further proof is that he might easily have done so had he wished. His account of number begins with a distinction between Unity, being, and difference; but he has just analyzed Unity on a different principle, as containing its own being and unity as parts, and argued that since each of those parts must be and be one, ad infinitum, Unity is infinitely divisible (142C–43A). The effect of this, combined with his account of number, is to make Unity infinite in two senses: it is a successive infinite, as containing parts corresponding to the numbers, since each of the numbers is and is one (144D–E); and it is a continuous infinite, as being divisible into parts which are infinitely divisible.[10] Clearly, infinite divisibility provides a basis for an infinite plurality of units. As Aristotle remarks (*Phys.* 3. 206b3–10, trans. Hardie): "In a way the infinite by addition is the same thing as the infinite by division. In a finite magnitude, the infinite by addition comes about in a way inverse to that of the other. For in proportion as we see division going on, in the same proportion we see addition being made to what is already marked off. For if we take a determinate part of a finite magnitude and add another part determined by the same ratio (not taking in the same amount of the original whole), and so on, we shall not traverse the given magnitude." That is, what is infinitely divisible is infinitely numerable; the series of numbers corresponds to the series of divisions.[11] Parmenides, then, since he has already shown that Unity is infinitely divisible, might easily demonstrate the existence of a plurality of units corresponding to every number. He does not do so.

The notion of number is ambiguous. Aristotle, after concluding in the *Physics* that Time is a kind of number, goes on to remark (4.219b5–8): "Number, we must note, is

9. Which Greek number theory regarded merely as abbreviated addition; see *Elements* 7, Def. 15, and Heath, *op. cit.*, p. 287.

10. This is one of Aristotle's two definitions of continuity: see *Phys.* 6.232b24–25, 231b15–16, 1.185b10, *Cael.* 1.268a6–7. For the second definition, in terms of identity of touching extremities, see *Phys.* 5.227a10–15. For numbers as successive and not continuous, see *Phys.* 5.227a20, *Metaph.* 13.1085a4. Because Greek mathematics lacked both real and rational numbers, there was no concept of a number continuum, nor, therefore, of a number line.

11. T. L. Heath in *Mathematics in Aristotle* (Oxford, 1959), pp. 106, 108–9, translated this passage in such a way as to imply, not the numbering of parts according to divisions, but an infinite convergent series whose sum is an *arithmetical* limit toward which the series tends, such as, though Aristotle's text does not specify a rational ratio, $1/2 + 1/4 + 1/8 + \cdots + 1/2^n + \cdots = 1$. But this conception, though it has an analogue in Greek geometry in the method of exhaustion and in successive rational approximations to irrational ratios, is unexampled in Greek arithmetic. The reason for this, presumably, is lack of rational and real numbers, as distinct from rational and irrational ratios; an infinite series can sum at a line, but not at a number. A. E. Taylor (*Plato* [London, 1960], p. 501, cf. pp. 509–11); for further discussion, see also Taylor's article in *Mind*, XXXV (1926), 419–40, XXXVI (1927), 12–33; see also D. W. Thompson, *Mind*, XXXVIII (1929), 43–55, who claimed to find the real numbers at *Epinomis* 990C–991B; but that text, which limits numbers to the odd and even, does not support him. It is, then, unlikely that Aristotle understood his "infinite by addition" to include infinite convergent series; see M. Claggett, "Novel Trends in Science," *Art, Science and History in the Renaissance* (ed. C. S. Singleton) (Baltimore, 1968), pp. 296–97.

used in two senses—both of what is counted or the countable and also of that with which we count. Time obviously is what is counted, not that with which we count: these are different kinds of thing." Parmenides, if he were a Platonist (cf. 135B–C), would suppose that that with which—or by which—we count are Idea Numbers or Forms, such as Twoness, Threeness, and so on. These Forms are not pluralities of units, but the number properties of such pluralities: Twoness is the Form common to all pairs, Threeness the common characteristic of all triples. Because "Number" Forms are not pluralities of units, they are "Inaddible"; Twoness plus Twoness amounts neither to Fourness nor four. Aristotle's testimony that, according to Plato, "there is a first two and first three, and the numbers are not addible to each other,"[12] is amply supported by the *Phaedo.*[13] Number Forms, of course, cannot be generated by arithmetical operations, since those operations, which involve counting, presuppose their existence; being Forms, they cannot, in fact, be generated by anything at all. Nor is there anything in Parmenides' argument which suggests otherwise. It is unlikely, then, that Parmenides is attempting to generate number in either of its two senses, as pluralities of units or properties of pluralities.

The assumption that Parmenides is attempting to generate number is largely based on the notion that he generates 1, 2, and 3 by counting. As Cornford put it, in attempting to explain how generation by multiplication could generate primes, "Plato evidently includes addition and starts with that when he *adds* one term to another to make two, and two to one to make three."[14] But examination of 143C–D shows this to be inaccurate. The argument does not proceed by counting, but is linguistic; and arithmetical propositions, so far from being proved by counting, are presupposed. Parmenides remarks that it is possible to mention (*eipein*) unity and to mention being, and thus "each of two" has been mentioned. But the English "two" is more explicit than the text, which contains only the genitive dual *autoin* (143C6). It is from use of this peculiarity in the inflection of his language—the existence of a dual, besides a singular and plural—that Parmenides goes on to infer that *both* have been mentioned, and that things cannot be both unless they are *two*. He then infers that for whatever is two, each of the two must be one.

It will be observed that, so far from adding one to one to make two, Parmenides has simply observed that the use of the dual implies two, and that two implies two ones. There is no suggestion in this that the number two consists in two units, any more than the dual consists in two units; nor is there any suggestion that the number two has been generated out of units, any more than the dual has been generated out of units. His inference that each of two is one is merely an application of the truth that $2 = 2 \times 1$ (see 143E2): Parmenides, instead of generating numbers, is merely using arithmetic.

The same is true of the putative generation of three. Parmenides argues that if each member of the pairs he has listed—unity and being, unity and difference, difference and being—is one, then when any one whatever is added to any pair whatever, the sum produced is three (*tria gignetai ta panta*, 143D7). This of course does not generate the number three, either as a property or a plurality of units; it is merely an application of the truth that $2 + 1 = 3$.

Parmenides next infers that, if the existence of pluralities having two and three members is granted, the existence of all numbers follows. If there are two things, there is twice, since $2 = 2 \times 1$; if there are three things there is thrice, since $3 = 3 \times 1$. Therefore there is thrice two and twice three: $3 \times 2 = 2 \times 3 = 6$. Therefore there are even-times even numbers, odd-times odd numbers, and odd-times even and even-times odd numbers.[15] There is, then,

12. *Metaph.* 13.1083a32.

13. See *Phaedo* 96E–97B, 101B–C, 104A–B, and H. F. Cherniss, *The Riddle of the Early Academy* (Berkeley and Los Angeles, 1945), pp. 33–37; *Aristotle's Criticism of Plato and the Academy*, pp. 300–304 and appendix vi; C. Wilson, *CR*, XVIII (1904), 247–60; W. D. Ross, *Plato's Theory of Ideas* (Oxford, 1951), pp. 180–81.

14. *Op. cit.*, p. 141, n. 2, italics Cornford's.

15. This is a standard classification of number in Greek arithmetic. See *Elements* 7, Defs. 8–10, and Heath, *op. cit.*, II, 281–84, and *History of Greek Mathematics*, I (Oxford

no number remaining which does necessarily exist. And if number is, there is an unlimited multitude of things which are, since number is unlimited in multitude.

The necessity involved in the claim that, by this procedure, there is no number which does not necessarily exist, is logical rather than ontological. Parmenides' argument is based on what we should now call closure: if *a* and *b* are integers, the product and sum of *a* and *b* are integers.[16] Given this as an axiom, and the existence of 1, 2, and 3 as integers,[17] the existence of all other integers follows.

An existence proof, of course, is not a generation. Parmenides has provided no indication that any number or numbers can be constructed or derived from simpler constituents. His argument is compatible with the view that numbers are timeless objects which no more admit of generation than they admit of destruction; it is also compatible with the view that numbers are simple essences incapable of analysis into ontologically (as distinct from numerically) prior and posterior elements. In short, Parmenides' account is compatible with the assumption that numbers are Forms or Ideas. Given that assumption, the role of an existence proof is merely that of a *ratio cognoscendi*.

If the foregoing interpretation is sound, the force of Parmenides' argument is hypothetical. If Unity exists, then Unity, being, and difference exist. There is a plurality, then, with those members; but to recognize the existence of that plurality is to commit oneself to the truths that $2=2\times 1$ and $2+1=3$. To accept any mathematical truth is to accept every mathematical truth; if any number exists, every number exists. Therefore, the existence of a plurality with three members implies the existence of a plurality with infinitely many members, namely, the plurality of numbers.

This argument, of course, does not prove the truth of arithmetic; it assumes it. Nor does it analyze what number, in itself, is: Parmenides' argument is silent on the question of whether numbers are pluralities of units, or whether they are Forms, or whether they are, perhaps, "intermediates," as Aristotle testifies.[18] The effect of Parmenides' argument is

1921), 71–72. It is not clear that the classification is exhaustive, for it is uncertain whether it classifies the primes; since Plato supposed that 1 is odd (see *Hip. maj.* 302A), the primes may have been classified as odd-times odd numbers, though this would have produced problems for even numbers: $4 = 4\times 1$. (This may explain the tradition which Aristotle preserves that 1 is *both* odd and even.) The classification is certainly not exclusive, since some numbers are both even-times even and even-times odd; thus $12=2\times 6=4\times 3$. There was, however, an ancient tradition that even-times even numbers are always (in effect) of the form 2^n.

It is interesting to note the connection of this classification with the theory of proportion. Parmenides has already introduced 1, the measure of number; 2, the first even number; and 3, the first odd number (1 being odd but not a number). His classification introduces 4, the first even-times even number; 6, the first even-times odd or odd-times even number; and 9, the first odd-times odd number (unless the primes 5 and 7 may be so classified). These terms stand in the three main proportions described by Archytas of Tarentum in a fragment of his lost work *On Music* preserved by Porphyry (*In Ptol. harm.* 93. 5 [Düring]; Diels-Kranz, *Fragmente der Vorsokratiker*[6] [Berlin, 1951], Archytas, B 2; cf. Heath, *History of Greek Mathematics*, I, 85–86). 1, 2, 4, and 4, 6, 9 are in geometric proportion ($a:b=b:c$); 1, 2, 3, and 2, 4, 6 are in arithmetic proportion ($a-b:b-c=a:a=b:b=c:c$; 2, 3, 6 are in harmonic proportion ($a-b:b-c=a:c$). In the *Timaeus* (31C) continued geometric is said to be the most perfect of bonds, and Greek number theory took it to be the most perfect and primary proportion (cf. Cornford, *Plato's Cosmology* [London, 1938], p. 45, citing Adrastus on the authority of Theon). It is for that reason that the Demiurge uses it in the construction of the World's body, using the theorem that two mean proportionals are required to connect cubics, and thus solids, whereas only one mean is required to connect squares, and thus planes (32A–B). The proportions in question are $p^2:pq=pq:q^2$ and $p^3:p^2q=p^2q:pq^2=pq^2:q^3$. The first proportion is satisfied by the first numbers in the classification Parmenides has provided: 6 is the mean proportion to $4=2^2$ and $9=3^2$. The second proportion, which Nicomachus called "a Platonic theorem" (*Arith.* 2. 24. 6 [D'Ooge]), requires $8=2^3$ and $27=3^3$, i.e., twice twice two and thrice thrice three. It is cited by Theon (*Plat. Arith.* 94. 11–14 [Hiller]) as the "second tetractys," subsequent to the first or Pythagorean tetractys summing at 10.

16. The objection that Parmenides' method does not generate the primes (cf. Ar. *Metaph.* 1.987b33 and Ross, *Plato's Theory of Ideas*, p. 187) neglects the fact that multiplication is abbreviated addition. It is not, however, clear that his account classifies the primes; but that, of course, is a different thing.

17. Strictly, it would seem that only the existence of 1 need be assumed, granting the operations of addition and (subsequently) multiplication. Parmenides, however, appears to assume that these operations presuppose duality; this strongly suggests that number is prior to the operations of addition and multiplication, in that these operations do not enter into the definition of number, or constitute its essence, but are necessary properties of number. Cf. *Phaedo* 101B–C.

18. *Metaph.* 1 987b14–18, cf. 7. 1023b19. For further discussion see Ross, *Aristotle's "Metaphysics,"* I, 166–68. Aristotle's account in fact identifies numbers and Ideas (cf. Ross, *op. cit.*, p. 164), but posits intermediates which differ from sensibles in being eternal and unchangeable, and from Ideas in that there are many alike, whereas each Idea is unique

simply to establish the implication that Proclus claims for it: "If One exists, number will exist, from which it follows that (infinite) plurality exists."[19] If it is true that unity, being, and difference exist if anything exists, the effect is still broader; Parmenides' argument provides an explanation of the Eleatic Stranger's remark in the *Sophist* (238A–B) that number must exist if anything exists; and it helps to explain why, in the *Theaetetus* (185D), numbers are listed along with unity, existence and nonexistence, likeness and unlikeness, and sameness and difference, as Common Terms which the mind is capable of contemplating through its own activity apart from the organs of sense.

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This interpretation is offered, presumably, in order to account for arithmetical operations: 2 + 2 = 4 uses two 2's, and Twoness itself is inaddible. This implies a distinction between mathematical number and Number Forms. But although Aristotle's testimony on this point is often accepted as primary evidence for Plato's views (most recently by A. Wedberg, *Plato's Philosophy of Mathematics* [Stockholm, 1955]), it is almost certainly mistaken. See Cherniss, *Riddle of the Early Academy*, pp. 33–37, and P. Shorey, *The Unity of Plato's Thought* (Chicago, 1903), pp. 82–85; see also *CP*, XXII (1927), 213–18, for Shorey's criticism of Adam's treatment of intermediates in *The "Republic" of Plato*, II (Cambridge, 1902; 2d ed., Rees, 1963), pp. 159–63.

19. *Schol. in Parm.* 1261. 18–21, Cousin.

THE UNITY AND SCOPE OF JUVENAL'S FOURTEENTH SATIRE

Recent interpretations of Juvenal's fourteenth Satire have dealt either with the substance of the poem, or with the structure, but have not considered substance and structure together.[1] Consequently the poem has been found difficult, and interpretation has been an exegesis on the stereotypes of satire. But Juvenal, it seems, intended more than a mere restatement of conventional topics. The poem is not a catalogue of Rome's vices, with particular emphasis on *auaritia*; rather it is a statement of Juvenal's traditionalism and comprehensive pessimism, suitably illustrated with examples of vice or virtue in its successive parts.

The Satire begins with the slashing accusation that it is the parents who corrupt their children. This is addressed to one Fuscinus, unknown to us and perhaps fictitious because his identity soon merges with that of the reader.[2] The accusation is shocking because it is a sweeping one and boldly phrased: "Plurima sunt, Fuscine, et fama digna sinistra / et nitidis maculam haesuram figentia rebus" (1–2), but its greatest impact comes from the *para prosdokian* ending of the sentence, with *parentes* (3) set emphatically at the end of the line: the guardians of the *mos maiorum* are the sources of corruption. Now the reader is prepared for an exposé of vice and decay at the core of Roman life. Still at a tender age, the scion of a good family,[3] perhaps a future consul, gambles at dice with his father (4–5). Another young man disappoints his family and relations; he withstands even the influence of his teachers; and under the instruction of a spendthrift parent devotes himself to gluttony (6–14). Moderation and humanity disappear because of the fanatic cruelty of a father (15–24). And chastity vanishes for want of a teacher (25–30). After a long philosophical digression about human weakness and proper aspirations intruded in the middle of the catalogue (31–85), we are introduced to a father, who, as if to disprove the possibility of righteousness, squanders his fortune in building villas, and who is followed by a yet more excessive son who destroys the material basis of his respectability in more extravagant architectural spending (86–95). And yet more seriously, the Judaizing tendencies of a father, followed by the conversion of his son,[4] undermine the Roman religious tradition (96–106).

1. Cf. J. A. Gylling, *De argumenti dispositione in satiris IX–XVI Iuvenalis* (Lund, 1889); V. D'Agostino, "La satira XIV di Giovenale," *Convivium*, IV (1932), 227–44; G. Highet, *Juvenal the Satirist* (Oxford, 1954), chap. xxiv and notes; E. N. O'Neil, "The Structure of Juvenal's Fourteenth Satire," *CP*, LV (1960), 251–53. The text quoted is taken from *A. Persi Flacci et D. Iuni Iuuenalis saturae*, ed. W. V. Clausen (Oxford, 1959).

2. Fuscinus is addressed by name only once in the first line of the poem. Elsewhere, more than sixty times, Juvenal uses the second person singular of verbs and pronouns instead. Fuscinus is quickly forgotten, and the reader feels addressed himself.

3. Cf. Mau in *RE*, III (1899), 1048 ff., *s.v.* "Bulla" (2).

4. Cf. G. Highet (above, n. 1), p. 283.